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SOME RESULTS OF THE IMPLEMENTATION OF THE "PODMODELI" PROGRAM FOR THE GAS DYNAMICS EQUATIONS†

L. V. OVSYANNIKOV

Novosibirsk

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The results of investigations using the PODMODELI (SUBMODELS) program, which is aimed at the exhaustive use of the symmetry of the gas dynamics equations to construct classes of exact solutions (submodels) of these equations, are summarized. The starting point is the fact that any Lie group of transformations of the basis space (of all independent and dependent variables) which is admitted by the gas dynamics equations can generate certain submodels. In order to describe the infinite set of submodels which arises from this in a compact manner, a number of techniques for ordering them is proposed: with respect to the equation of state of a gas, using a similarity criterion, according to their types (rank, defect), with respect to the property of evolutionarity and according to the criterion of regularity. Examples of new submodels are presented. The most significant work carried out using the PODMODELI program up to the present time is indicated in a list of references. © 1999 Elsevier Science Ltd. All rights reserved.

Suppose a system E of differential equations for the required functions u of the independent variables x is given. The idea behind the PODMODELI (SUBMODELS) program is associated with the property of the symmetry of the system E and, in fact, its invariance with respect to a Lie group of transformations of the basis space (the sets of all x, u). The final aim of the PODMODELI program lies in exhausting all possibilities which are made available by this property to construct the exact solutions of system E.

This paper is concerned with the presentation of certain results of the implementation of this program in the case of the gas dynamics equations.

1. OBJECT OF THE INVESTIGATION

The system of differential equations of gas dynamics

$$\rho D\mathbf{u} + \nabla p = 0, \quad D\rho + \rho\,\mathrm{div}\,\mathbf{u} = 0, \quad DS = 0 \tag{1.1}$$

is considered with the *equation of state*

$$p = F(\rho, S) \tag{1.2}$$

defined in the basis space $R^{10}(t, \mathbf{x}, \mathbf{u}, \rho, p, S)$ with a time t, coordinates $\mathbf{x} = (x, y, z) = (x^1, x^2, x^3)$, velocity vector $\mathbf{u} = (u, v, w) = (u^1, u^2, u^3)$, density ρ, pressure p and entropy S. The total differentiation operator $D = \partial_t + \mathbf{u}\nabla$, $\nabla = (\partial_x, \partial_y, \partial_z)$. It is assumed that the gas is *normal*, that is, the function F in (1.2) is such that $F > 0$, $F_\rho > 0$, $F_S > 0$. The speed of sound $c > 0$ is determined by the relation $c^2 = F_\rho(\rho, S)$.

The *property of symmetry* of the system of gas dynamics equations lies in the fact that the system is invariant with respect to the 11-parameter Lie group, G_{11} of transformations of the basis space (it is also said that this system *admits* the group G_{11}), which is conveniently written in terms of the corresponding Lie *algebra* L_{11} of *operators* acting in this space. The basis in L_{11} (the operators $X_1, X_2, \ldots, X_{11}$) is chosen as

$$X_i = \partial_{x^i}, \quad X_{i+3} = t\partial_{x^i} + \partial_{u^i}, \quad X_{i+6} = \varepsilon_{ij}^k (x^j \partial_{x^k} + u^j \partial_{u^k}) \quad (i = 1, 2, 3)$$
$$X_{10} = \partial_t, \quad X_{11} = t\partial_t + x^i \partial_{x^i} \tag{1.3}$$

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where $\partial_t = \partial/\partial t$, $\partial_{xi} = \partial/\partial x^i$, etc., ε^k_{ij} is a standard skew-symmetric tensor with $\varepsilon^3_{12} = 1$ and it is assumed that summation is carried out over repeated indices [1, 2].

Remark. The entropy S is determined, apart from the substitution $S \to \varphi(S)$, with an arbitrary, non-constant function φ. This transformation of the entropy, which is an equivalence transformation for the equation of state and is always admitted by the gas dynamics equations, is only taken into account below in writing the equation of state.

In the case of an equation of state (1.2) of general form, the algebra L_{11} is the *widest* Lie algebra of operators which are admitted by the system of gas dynamics equations.

2. CLASSIFICATION ACCORDING TO THE EQUATION OF STATE

Depending on the form of the equation of state (1.2), 13 cases of *extension* of the admissible Lie algebra are distinguished. In particular, in the case of a polytropic gas with an equation of state $p = S\rho^\gamma$, the Lie algebra L_{11} is extended to L_{13} by means of the operators

$$X_{13} = t\partial_t - u^i\partial_{u^i} + 2\rho\partial_\rho, \quad X_{14} = \rho\partial_\rho + p\partial_p \tag{2.1}$$

for an arbitrary value of the adiabatic index γ. If, however, $\gamma = 5/3$, then a further operator

$$X_{12} = t^2\partial_t + tx^i\partial_{x^i} + (x^i - tu^i)\partial_{u^i} - 3t\rho\partial_\rho - 5tp\partial_p \tag{2.2}$$

which extends L_{13} to L_{14}, is added to the operators (2.1). A complete list of all extensions is given in [1].

In the analysis of the submodels obtained it turns out to be advisable to pick out the *special models* of the gas motions which are indicated below, together with the corresponding systems of differential equations.

Isentropic models. ($S = S_0 = \text{const}$) with an equation of state $p = F(\rho, S_0)$

$$D\mathbf{u} + a(\rho)\nabla\rho = 0, \quad D\rho + \rho\,\mathrm{div}\,\mathbf{u} = 0 \tag{2.3}$$

where $a(\rho) = c^2/\rho$.

Barochronous models. ($p = p(t)$) with an equation of state $p(t) = F(\rho, S)$

$$D\mathbf{u} = 0, \quad \mathrm{div}\,\mathbf{u} = a(t, S), \quad DS = 0 \tag{2.4}$$

where $a(t, S) = -p'(t)/\rho c^2$. In the case of a polytropic gas $a = a(t) = -p'/\gamma p$. *Isobaric models* are a special case of barochronous models for which $p = \text{const}$, $a = 0$.

Thermal models. ($\rho = \rho_0 = \text{const} > 0$) with an equation of state $p = F(\rho_0, S)$.

$$D\mathbf{u} + \nabla p = 0, \quad \mathrm{div}\,\mathbf{u} = 0, \quad Dp = 0 \tag{2.5}$$

Isothermal models. ($c = \text{const}$) with an equation of state (1.2). If $F_{\rho S} \neq 0$, then system (1.1) reduces to (2.5). In the case when $F_{\rho S} = 0$, one of the classification versions of the extension of the admitted group described in [1] arises.

Special models, of course, admit of the group G_{11}. However, with the exception of the isentropic models, the widest groups which are admitted by these models have still not been calculated and even system (2.5) has not been reduced to an involution up to this point.

3. CONSTRUCTION OF SUBMODELS

Each submodel is generated by a certain subgroup H of an admitted group and, in this case, it is called an *H-submodel.*

We recall the general procedure for constructing H-submodels in the case of an admitted group H of the system of equations E specified in the basis space $R^{n+m}(x, u)$ with the required functions

$u = (u^1, \ldots, u^m)$ of the independent variables $x = (x^1, \ldots, x^n)$. Suppose $I = (I^1, \ldots, I^l)$ is the complete set of functionally independent *invariants* $I^j = I^j(x, u)$ $(j = 1, \ldots, l)$ of the group H (its *universal invariant*) and suppose $R^l(I)$ is the *space of the invariants*. A manifold M, specified by a certain number $\mu > 0$ of independent scalar equations, is picked out in $R^l(I)$. Here, it is necessary that $\mu \leqslant m$. The dimension of M in $R^l(I)$ is equal to $\sigma = l - \mu$. In the space $R^{n+m}(x, u)$, the manifold M has a dimension $n + m - \mu = n + \delta$ which leads to the relation

$$\sigma = \delta + l - m \tag{3.1}$$

In order to satisfy these conditions, the subsets $u' = (u^1, \ldots, u^{m-\delta})$, $I' = (I^1, \ldots, I^{m-\delta})$ are found for which

$$\det(\partial I'/\partial u') \neq 0$$

and the σ of invariants of the subset $I'' = (I^{m-\delta+1}, \ldots, I^l)$ is independent of the quantities u'. Then, using the variables

$$v = I'(x, u), \quad y = I''(x', u'') \tag{3.2}$$

the equations of the manifold M can be written in the form

$$v = V(y) \tag{3.3}$$

The quantities $u'' = (u^{m-\delta+1}, \ldots, u^m)$ occurring in (3.2) are called *"superfluous" functions*. As a result of substituting the expressions for u' obtained from (3.3) into the system E, a *factor system* E/H is obtained which decomposes into two subsystems: an *invariant* subsystem for the required functions $V(y)$ and an additional, generally speaking, *overdetermined* subsystem for the superfluous functions u''. The number σ, which is equal to the number of independent variables in the invariant subsystem, is called the *rank* and the number δ, which is equal to the number of "superfluous" functions u'', is called the *defect* of the submodel.

The factor system E/H is called a *partly invariant submodel* of type (σ, δ) and is also denoted by $H(\sigma, \delta)$. It is important that one and the same group H can generate partly invariant submodels of several different types.

The solutions of the equations of a submodel $H(\sigma, \delta)$ are called *partly invariant solutions* (and, when $\delta = 0$, *invariant solutions*) of system E.

A submodel $H(\sigma, \delta)$ is said to be *regular* if the variables y in (3.2) are independent of u''. Otherwise, it is said to be irregular [3].

In applications, instead of a group H, the Lie algebra L of the operators over the base space $R^{n+m}(x, u)$ corresponding to this group is considered. Suppose L has a dimension r and a basis of operators

$$H_\alpha = \xi^i_\alpha(x,u)\partial_{x^i} + \eta^k_\alpha(x,u)\partial_{u^k} \quad (\alpha = 1, \ldots, r) \tag{3.4}$$

The dimension l of the universal invariant is defined in terms of a $r \times (n + m)$ matrix of the coordinates of the operators (3.4). If r^* is the overall rank of this matrix, which is describable by the formula

$$r_* = \text{o. r. } (\xi^i_\alpha, \eta^k_\alpha) \tag{3.5}$$

then $l = n + m - r_*$, and relation (3.1) takes the form

$$\sigma = \delta + n - r_* \tag{3.6}$$

The defect δ must satisfy the inequalities

$$\max\{r_* - n,\ m - q,\ 0\}, \leqslant \delta \leqslant \min\{r_* - 1,\ m - 1\} \tag{3.7}$$

which follow from the preceding construction, where q is the overall rank of the Jacobian matrix $\partial I/\partial u$.

For example, the whole algebra L_{11} with the basis (1.3) can be taken as L for the gas dynamics equations ($n = 4, m = 5$). In this case, formula (3.5) gives $r_* = 7$ and $I = (\rho, p)$, that is, $q = 2$. Inequalities (3.7) take the form $3 \leqslant \delta \leqslant 4$, and it follows from (3.6) that $\sigma = \delta - 3$. Hence, L_{11} can generate submodels of types (0, 3) and (1, 4). It is not difficult to show that an isobaric model (2.4) is obtained as a partly invariant submodel of type (0, 3) and either an isentropic model (2.3) or a thermal model (2.5) is obtained as a partly invariant submodel of type (1.4).

4. OPTIMAL SYSTEMS OF SUBALGEBRAS

The implementation of the PODMODELI program for the gas dynamics equations involves taking account of all the admissible subalgebras, since each of them can produce some submodels. Moreover, the Lie algebra L_{11} contains infinitely many subalgebras. It has been pointed out [2] that not all the submodels obtained by this route are substantially different. Two submodels are said to be *similar* if one of them is obtained from the other by a certain reversible replacement of variables. It is found that this property is equivalent to the fact that the generating subalgebras are associated in an enveloping Lie algebra by the action of an internal automorphism of this algebra.

The set of classes of associated subalgebras of a given Lie algebra L is called the *optimal system* of subalgebras and is denoted by ΘL. The actual classes are identified by their *representatives* from which ΘL is made up. Actually, there are also infinitely many classes but their representatives can be combined into convenient series containing a small number of parameters, and ΘL is actually the list of all such series, each of which is assumed to be a single representative.

A sufficiently well tested algorithm [4] has been developed for calculating the optimal systems of subalgebras of finite dimensional Lie algebras. The optimal system ΘL_{11} for the Lie algebra L_{11} with the basis (1.6) consisting of 220 representative [1, Table 6] was calculated for the first time using this algorithm. The optimal systems ΘL_{13} [5] consisting of 1342 representatives and ΘL_{14} [6] consisting of 1826 representatives were also calculated.

A barely visible large collection (thousands) of possible, dissimilar submodels is thereby obtained for the gas dynamics equations. The new problem of searching for additional criteria for the ordering of this collection arose.

5. INVARIANT SUBMODELS

Submodels of the type $(\sigma, 0)$ describe invariant solutions of the gas dynamics equations. By virtue of (3.6) and (3.7), they generate subalgebras with $r_* \leqslant 4$ and have a rank $\sigma = 4 - r_*$. In the case of these submodels, the factor system reduces to a *determined* invariant subsystem, located in the involution. The equations of the factor system associate the partial derivatives of the required invariants with respect to three ($\sigma = 3$) or two ($\sigma = 2$) independent variables or they form a system of ordinary differential equations ($\sigma = 1$).

All submodels of the type $(\sigma, 0)$ $(\sigma > 0)$ are distributed over two classes: of *evolutionary form* (E) or of *stationary form* (S). Class (E) is generated by those subalgebras for which the time t is an invariant while class (S) is generated by subalgebras for which t is not an invariant. The number of invariant submodels of different ranks $\sigma = 3, 2, 1$ is shown in Table 1 for an equation of state of general form (GES) and for a polytropic gas (PG) with arbitrary γ.

The actual form of writing the equations of a submodel which is generated by a subalgebra H depends on the choice of the invariants I in the representation of solution (3.3). It has been noted in [7] and subsequently rigorously proved in [8] that, with a suitable choice of invariants, the factor system for all invariant submodels of ranks $\sigma = 1, 2, 3$ can be written in the same *canonical form* (which is different for classes (E) and (S)) and, in fact,

$$RD'\mathbf{U} + \mathbf{B}\nabla' P = \mathbf{f}, \quad D'R + R\,\mathrm{div}'\,\mathbf{U} = g, \quad D'S' = h \tag{5.1}$$

with invariant velocity vectors $\mathbf{U} = (U^1, U^2, U^3)$, density R, pressure P and entropy S' which are functions of the invariant independent variables in class (E) and $(y^1, \ldots, y^\sigma)$ in class (S). In the case of a general equation of state, this equation is $P = F(R, S')$ with the function F (1.2) while, for a polytropic gas, it has the form $P = S'R^\gamma$. The differential operations of the gradient ∇', of total differentiation D' and divergence div$'$ solely act with respect to the corresponding independent variables For example, when $\sigma = 2$, these operations have the form

$$\nabla' = (\partial_{y^1}, 0, 0), \quad D' = \partial_t + U^1\partial_{y^1}, \quad \mathrm{div}'\,\mathbf{U} = U^1_{y^1}$$

in class (E) and

$$\nabla' = (\partial_{y^1}, \partial_{y^2}, 0), \quad D' = U^1\partial_{y^1} + U^2\partial_{y^2}, \quad \mathrm{div}'\,\mathbf{U} = U^1_{y^1} + U^2_{y^2}$$

in class (S).

Table 1

σ	3		2		1	
	GES	PG	GES	PG	GES	PG
(E)	6	11	10	28	18	50
(S)	7	18	16	73	21	124

The elements of the 3×3 matrix

$$\mathbf{B} = \begin{Vmatrix} B^{11} & 0 & 0 \\ 0 & B^{22} & B^{23} \\ 0 & B^{32} & B^{33} \end{Vmatrix}$$

are solely functions of the invariant independent variables which depend on the submodel. The right-hand sides in (5.1) are functions of the invariants which are specific for each submodel and do not contain derivatives of the required functions.

Note that not all of the submodels of rank 3 for a general equation of state presented in [1] are written in canonical form.

It has also been established that all the invariants of a submodel have an equivalent representation in the form of a system of *invariant integral laws of conservation* of mass, momentum and energy [9].

6. SUBMODELS OF THE TYPE (0, 0)

This type of invariant submodel is distinguished by the fact that the required invariants must be constants. Hence, the factor system which relates these constants consists of algebraic (and not differential) equations. The solutions of the gas dynamics equations, which are described by submodels of the type (0, 0), are called *simple solutions*. A constant solution belongs to such solutions, for example. Simple solutions are generated by four dimensional subalgebras L_4 with the number (3.5) $r_* = 4$.

In the case of a general equation of state, all the simple solutions describe isobaric motions of a gas (2.4). In the case of a polytropic gas, out of the 290 representatives of L_4 which are contained in ΘL_{13}, 85 simple solutions are found to be generating and only 34 of them do not refer to the special gas motions (2.3)–(2.5). In the complete list of these 34 solutions which has been compiled, those solutions which depend substantially on four independent variables have been separated out. There are just eight non-similar submodels and they can be represented in the form of three subclasses [10].

For example, one of the simple solutions for a polytropic gas with $\gamma = 3$ in polar coordinates r, θ ($y = r\cos\theta$, $z = r\sin\theta$) with components of the velocity vector u (along the x axis), v_r (the radial component) and v_θ (the peripheral component) is given by the formulae

$$u = -\frac{1}{1-t}(x + Ur), \quad v_r = 0, \quad v_\theta = \frac{r}{1-t}W \tag{6.1}$$

$$\rho = (1-t)^{-A-1} r^{A-2} e^{A\theta/W} R, \quad p = (1-t)^{-A-3} r^{A} e^{A\theta/W} P$$

where U, W and A are arbitrary constants and $P/R = W^2/A$, $A > 0$. The phenomenon of a *collapse of the density* (and pressure) is characteristic of this (and many other simple solutions). The trajectory of a gas particle which has "started out" at $t = 0$ from a point (x_0, r_0, θ_0) with the values ρ_0, p_0 is given by the equations

$$x = (x_0 + Ur_0)(1-t) - Ur_0, \quad r = r_0, \quad \theta = \theta_0 + W\ln(1-t) \tag{6.2}$$

which follow from (6.1), and this particle "bears" the values $\rho = (1-t)^{-1}\rho_0, p = (1-t)^{-3}p_0$. The collapse occurs as $t \to 1$ and, then, $\rho \to \infty$ ($p \to \infty$). It follows from (6.2) that the *collapse manifold* is the cone $x = -Ur$. All the gas particles which have "started out" at $t = 0$ from any point of a cylinder with $r = r_0$ and which remain on it when $t > 0$, executing an unbounded number of loops with an exponentially decaying step and a peripheral velocity which increases in an unbounded manner when $t \to 1$ tend towards the intersection of this cone with the cylinder.

7. SYMMETRY OF INVARIANT SUBMODELS

Each of the submodels mentioned above in Section 5 possesses a definite symmetry which can be calculated, regardless of the origin of the submodel, using the general algorithm for finding the Lie algebra of operators admitted by the system of differential equations [2]. Moreover, this symmetry is, at least partly, previously known by virtue of the following theorem [2]. Suppose a system E admits a Lie algebra L, the subalgebra $H \subset L$ generates an invariant submodel E/H and $\mathrm{Nor}_L H$ is the *normalizer* of the subalgebra H in L. Then, the factor system E/H admits the *factor algebra* $\mathrm{Nor}_L H/H$.

It is established by direct calculation that, in the case of the majority of invariant submodels of the gas dynamics equations, the widest admissible Lie algebra is a direct sum of a factor algebra $\mathrm{Nor}_L H/H$ and a certain infinite dimensional Lie algebra L_∞ which arises on account of the reduction in the dimensionality of the subspace of the independent variables in the factor system E/H.

The simplest example gives a submodel of the *two-dimensional* gas motions which is generated by the one dimensional subalgebra $H = \{X_3\}$ (transport along the z coordinate). Here, in the factor system, the component w of the velocity vector only occurs in the momentum equation $Dw = 0$ and the admissible operator $X_\varphi = \varphi(w)\partial_w$ with an arbitrary function $\varphi(w)$ is added to the normalizer.

A non-trivial example of such an extension gives a submodel of the *steady* gas motions subject to the condition that, in the equation of state (1.2), the function F depends solely on the product ρS, that is, for an equation of state of the form

$$p = F(\rho S) \tag{7.1}$$

In this case, L_∞ is generated by the operator (found by Yu. A. Churkunov and the author)

$$Z_\Phi = \Phi(S, B)(u^i\partial_{u^i} - 2\rho\partial_\rho) \tag{7.2}$$

where Φ is an arbitrary function of two variables and B is a Bernoulli function $B = |\mathbf{u}|^2 + 2i$ with a specific enthalpy $i = Sj(\rho S)$, where $j(\xi) = \int \xi^{-1} df(\xi)$. An infinite Lie pseudogroup of transformations $(\mathbf{u}, \rho, p, S, B) \to (\mathbf{u}', \rho', p', S', B')$ corresponds to the operator (7.2)

$$\mathbf{u}' = \varphi\mathbf{u}, \;\; \rho' = \varphi^{-2}\rho, \;\; p' = p, \;\; S' = \varphi^2 S, \;\; B' = \varphi^2 B \tag{7.3}$$

with an arbitrary function $\varphi = \varphi(S, B)$. The mapping (7.3) was pointed out for the first time by Munk and Prim [11]. It enables one to transform any continuous steady-state solution with an equation of state (7.2) into either an isentropic solution (S = const.) or an isodynamic solution (B = const.).

The calculation of the admissible operators in the case of invariant submodels of rank $\sigma = 3$ is helped considerably by the property of "x-autonomy" which is inherent in them: the coordinates of these operators accompanying derivatives with respect to the independent variables are *independent of the required functions*. This is established by the use of the sufficient criterion of "x-autonomy", which holds for the class of certain systems of first-order quasilinear equations [12].

By considering any submodel of the gas dynamics equations as the "initial" model without a known symmetry, it is possible to seek its submodels. They are called the *two-step* submodels for the gas dynamics equations (1.1). The question arises here as to whether these two-step models contain all of the submodels in the initial collection Ω

The answer is given by the "LOT lemma" [13] which holds for any system of differential equations E which admits a Lie algebra L. Suppose a subalgebra $H \subset L$ generates an invariant submodel E/H and let the subalgebra $H' \subset \mathrm{Nor}_L H/H$. Then the subalgebra $M \subset \mathrm{Nor}_L H'$, for which H' is the factor algebra M/H (M is the original of H' in the case of the homomorphism $\mathrm{Nor}_L H \to \mathrm{Nor}_L H/H$), is uniquely defined. The system E admits M and it is assumed that the invariant submodel E/M exists. In its turn, the factor system E/H admits a subalgebra $H' = M/H$ and it is assumed that the invariant submodel $(E/H)/H'$ exists, which is also a two-step submodel for E. The LOT lemma establishes the equivalence of these submodels, which is written in the form of a symbolic equality

$$(E/H)/(M/H) = E/M \tag{7.4}$$

Consequently, additional two-step submodels, which do not occur in the collection Ω, can only appear in the case when they are generated by subalgebras which contain elements of extensions to the Lie algebra which are admitted by the submodels from Ω.

8. PARTIALLY INVARIANT SUBMODELS

There are significantly more submodels of the type (σ, δ) with δ > 0 than invariant submodels in view of the fact that, for each subalgebra H, the defect δ can take several values (7.3) and, as has already been pointed out in Section 3, here there is no *a priori* constraint on the dimension of the generating subalgebra H.

The specific feature of partially invariant submodels lies in the fact that, in these submodels, a part of the factor system E/H is an *overdetermined subsystem* for the "superfluous" functions, and an analysis of its compatibility (its *reduction to an involution*) is required. Furthermore, in the case of a partially invariant submodel, its reduction to a *smaller defect* can occur. The phenomenon when a submodel $H(\sigma, \delta)$ proves to be simultaneously a submodel $H'(\sigma, \delta')$ of a generated subalgebra $H' \subset H$ of *the same rank* σ but of a *smaller defect* $\delta' < \delta$ is referred to by this term. In particular, reduction of a partially invariant submodel to an invariant submodel with $\delta' = 0$ is possible.

Experience is constructing partially invariant submodels show that, the greater the defect δ, the more difficult it is to reduce the factor system E/H to an involution. Hence, there is an urgent need for *a priori* tests for the reduction of a partially invariant submodel. At the present time, the following sufficient criterion of reduction is effectively used [2].

Suppose expressions of the form

$$\partial u^k / \partial x^i = f_i^k(x, u) \quad (i = 1, \ldots, n; k = 1, \ldots, m) \tag{8.1}$$

are obtained for a submodel $H(\sigma, \delta)$ when analysing the compatibility of the factor system E/H, using only algebraic operations and differentiation, for all derivatives of the required functions $u = (u^1, \ldots, u^m)$ with respect to all of the independent variables $x = (x^1, \ldots, x^n)$.

Then, any $H(\sigma, \delta)$ solution will also be a $H'(\sigma, 0)$ solution with respect to a certain subalgebra $H' \subset H$.

9. REGULAR PARTIALLY INVARIANT SUBMODELS

In the collection of partially invariant submodels of relative simplicity, the *regular* submodels $H(\sigma, \delta)$ are picked out in which all of the σ of the invariant independent variables in the factor system E/H depend solely on the initial independent variables. The separation of the regular partially invariant submodels is dictated by the fact that the number of these submodels for the gas dynamics equations is found to be completely visible, and the analysis of the compatibility of the overdetermined subsystems which arise is comparatively easy.

A complete listing of the 100 subalgebras which generate regular partially invariant submodels for the equations of gas dynamics in the case of a general equation of state (1.2) (a sample from ΘL_{11}) is given in [14]. Invariant submodels, which are always regular, are included in this list.

After eliminating invariant submodels from this list and, also, those which refer to the barochronous motions of the gas (2.4), 30 partially invariant submodels, which are not reduced, remain. All of them have been analysed. The description of submodels of types (3.1) and (2.2) has been published in [14], type (2.1) in [15] and types (1.2) and (1.1) in [16]. A similar list of regular partially invariant submodels for a polytropic gas is being compiled.

An example of a regular partially invariant submodel of type (1.2), which has not been published previously, is presented below. All the important factors in the analysis of regular partially invariant submodel are observed in this example.

Example. A submodel which is generated by the five-dimensional algebra $L_5 = \{X_2, X_3, X_5, X_6, X_{10}\}$ with the universal invariant $I = (x, u, \rho, S)$ is considered. For this submodel, the number (3.5) $r_* = 5$ and it follows from relations (3.6) and (3.7) that the regular partially invariant submodel which is generated by this L_5 must be of type (1.2).

Here, the solution is represented in the following way; the invariants u, ρ and S (and this also means the pressure p) must be functions solely of the coordinate x and the "superfluous" functions v, w can depend on all of the variables t, x, y, z. The factor system has the form

$$\rho u u_x + p_x = 0, \quad u S_x = 0, \quad u \rho_x + \rho(u_x + v_y + w_z)$$

$$Dv = 0, \quad Dw = 0 \quad (D = \partial_t + u\partial_x + v\partial_y + w\partial_z)$$

It follows from the third equation that the quantity $h = u_y + w_z$ is solely a function of the x coordinate. With the function $h = h(x)$, the factor system decomposes into the invariant subsystem

$$\rho u u_x + p_x = 0, \; u S_x = 0, \; u \rho_x + \rho u_x + \rho h = 0 \tag{9.1}$$

and the overdetermined subsystem for the "superfluous" v and w

$$Dv = 0, \; Dw = 0, \; v_y + w_z = h \tag{9.2}$$

It is further assumed that $u \neq 0$ (isobaric solutions are obtained differently).

In order to reduce system (9.2) to an involution, the operator D is applied to the third equation, which gives the relation $2(u_y w_z - u_z w_y) = u h_x + h^2$. Hence, $u_y w_z - u_z w_y$ depends solely on x, and the equation with the function $k = k(x)$

$$v_y w_z - v_z w_y = k \tag{9.3}$$

where $2k = u h_x + h^2$, is added to (9.2). Application of the operator D to (9.3) gives the closed relation $u k_x = -hk$. So, the functions h and k, which have been additionally introduced, must satisfy the system of equations

$$u h_x + h^2 = 2k, \; u k_x + hk = 0 \tag{9.4}$$

By virtue of Eqs (9.4), the system of four equations (9.2) and (9.3) for the required u, w is in the involution.

System (9.4) is integrated with the variable τ (the time of motion of the gas particles along the x axis) which is introduced by the equation

$$dx/d\tau = u(x) \tag{9.5}$$

and the general solution of system (9.4) is given by the formulae

$$h = Q'/Q, \; k = Q''/2Q, \; Q = a_0 + a_1\tau + a_2\tau^2 \tag{9.6}$$

where a_0, a_1, a_2 are arbitrary constants and primes indicate derivatives with respect to τ.

The invariant subsystem (9.1) is integrated in the form

$$u^2 + 2i(\rho) = 2b, \; \rho u Q = q, \; S = \text{const} \tag{9.7}$$

with $i(\rho) = \int \rho^{-1} dp$ and arbitrary constants b and q. Relations (9.5)–(9.7) determine (in implicit form) the required relations $u(x)$ and $\rho(x)$.

It remains to solve the overdetermined system (9.2), (9.3). In order to do this, the Lagrangian coordinate $\xi = t - \tau(x)$ is introduced with which this subsystem is integrated by linearizing it in the same way as was done in the case of the analogous overdetermined system, "canonical submodel of type (1.2)" in [14]. Finally, the solution is determined with an arbitrariness in two functions of a single argument.

The resulting gas motions are "combined" from the two components: a steady-state one-dimensional motion in the direction of the x axis (with velocity $u(x)$ and density $\rho(x)$) and a certain rather complex, transverse, unsteady motion in planes perpendicular to the x axis (with a velocity (u, w) which depends on t, ξ, y, z).

10. ON IRREGULAR PARTIALLY INVARIANT SUBMODELS

The possibility of constructing such submodels is significantly greater than the possibility of constructing regular submodels, since certain required functions also serve as independent variables in them. On the other hand, the analysis of the overdetermined systems which arise here is much more complex and frequently turns out to be almost insuperable. The *multiple waves*, which are generated by the subalgebra $L_4 = \{X_1, X_2, X_3, X_{10}\}$, where all of the required functions are invariants, serves as a classical example of irregular partially invariant submodels for the gas dynamics equations. Here, $r_* = 4$, and $\sigma = \delta$ is obtained from (3.6) and the inequality $0 \leqslant \delta \leqslant 3$ from (3.7). The type (0.0) gives a constant solution. A submodel of type (1.1) describes *simple waves*, a submodel of type (2.2) describes *double waves* and a submodel of type (3.3) describes *triple waves*. Of these, only the simple waves have been thoroughly studied. In particular, in the case of one-dimensional unsteady gas motions, these are Riemann waves and Prandtl–Mayer waves in the case of two-dimensional steady-state flows. As far as the double (and, even more so, the triple) waves are concerned, a complete description of these waves has still not been presented anywhere although a fairly large number of papers deal with various particular examples of such solutions [17, 18].

The principal difficulty lies in the fact that the reduction of the corresponding overdetermined systems to an involution requires quite high orders of *extension* of the systems of equations with partial derivatives with respect to four (or just three) independent variables. A tentative calculation shows the need for extension up to an order of greater than four, for which not only paper but also the memories of personal computers are not enough.

Matters are considerable simpler in the case of irregular partially invariant submodels of type (1.1), when the reduction to an involution reduces to calculating the commutators of first-order linear differential operators. One of the examples of such a partially invariant submodel is given in [19]. Four-dimensional subalgebras, for which $r_* = 4$, mainly serve as candidates for the generation of irregular partially invariant submodels of type (1.1). There is quite a large number of such subalgebras (see Section 6) and the discovery of new irregular partially invariant submodels continues at the present time.

11. ACTUAL SUBMODELS

The PODMODELI program envisages not only the study of the mathematical structures which arise for all submodels for the gas dynamics equations but also a more detailed representation of their physical content. In view of the abundance of submodels, such investigations have been carried out selectively up to now.

Submodels describing isobaric [20], two-dimensional [21], spiral [22], spiral barochronous [23], rotational [4] and general barochronous [25, 26] gas motions have been qualitatively investigated (group classification, first integrals, representation of the general solution, singularities in the solutions, etc.).

Individual submodels have been studied in greater detail. For instance, an example of a new regular partially invariant submodel of type (2.1), which is generated by a subalgebra of rotations, where the "superfluous" function is the angle made by the projection of the velocity vector onto a sphere with its meridians, is given in [27]. The construction of two-dimensional invariant (self-similar) solutions, which describe flows of a polytropic gas with close streamlines is considered in [28]. A non-trivial example of an irregular solution of type (1.1) for two-dimensional motions is given in [29]. A new invariant solution of type (1.0) has been investigated in [30]. Canonical forms of invariant submodels of rank two have been constructed for a general equation of state in [31].

12. CONCLUSION

The symmetry of the gas dynamics equations opens up the considerable possibility of discovering new gas motions which permit an exact description of the actual forms. The PODMODELI program, which is aimed at the systematic utilization of this possibility, has demonstrated its fruitfulness. During the course of its implementation, a series of computer programs have been developed and used, together with the development of analytical methods, in particular, for deriving systems of defining equations, calculating the normalizers of subalgebras and constructing canonical forms of submodels. Work using the PODMODELI program was initiated and is being continued by a group of researchers based at the M. A. Lavrent'yev Institute of Hydrodynamics of the Siberian Branch of the Russian Academy of Sciences. I wish to thank Yu. A. Chirkunov, A. A. Talyshev, S. V. Meleshko, S. V. Khabirov, A. P. Chupakhin, Ye. V. Mamontov, S. V. Golovin and A. A. Cherevko for the results of their investigations which they made available and have been used in this paper.

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REFERENCES

1. OVSYANNIKOV, L. V., The PODMODELI program. Gas dynamics. *Prikl. Mat. Mekh.*, 1994, **58**, 4, 30–55.
2. OVSYANNIKOV, L. V., *Group Analysis of Differential Equations*. Academic Press, New York, 1982.
3. OVSYANNIKOV, L. V., Regular and irregular partially invariant solutions. *Dokl. Ross. Akad. Nauk*, 1995, **343**, 2, 156–159.
4. OVSYANNIKOV, L. V., Optimal systems of subalgebras. *Dokl. Ross. Akad. Nauk*, 1993, **333**, 6, 702–704.
5. GOLOVIN, S. V., *Optimal system of subalgebras for a Lie algebra of operators admitted by the gas dynamics equations in the case of a polytropic gas*. Preprint No. 5–96. Inst. Gidrodinamiki, Íib. Otd. Ross. Akad. Nauk, Novosibirsk, 1996.
6. CHEREVKO, A. A., *Optimal system of subalgebras for the Lie algebra of operators admitted by the gas dynamics equations with an equations of state* $p = f(S)\rho^{5/3}$. Preprint No. 4–96. Inst. Gidrodinamiki, Íib. Otd. Ross. Akad. Nauk, Novosibirsk, 1996.
7. KHABIROV, S. V., The analysis of rank three invariant submodels of gas dynamics. *Dokl. Ross. Akad. Nauk*, 1995, **341**, 6, 764–766.
8. OVSYANNIKOV, L. V., *Canonical form of invariant submodels of gas dynamics*. Preprint No. 3–97. Inst. Gidrodinamiki, Sib. Otd. Ross. Akad. Nauk, Novosibirsk, 1997.
9. OVSYANNIKOV, L. V., Invariant integral laws of conservation. *Dokl. Ross Akad. Nauk*, 1996, **351**, 5, 599–602.
10. OVSYANNIKOV, L. V., On "simple" solutions of the equations of the dynamics of a polytropic gas. *Zh. Prikl. Mekh. Tekh. Fiz.*, 1999, **40**, 2, 5–12.
11. MUNK, M., PRIM, R., On the multiplicity of steady gas flows having the same streamline pattern. *Proc. Nat. Acad. Sci. USA.*, 1947, **33**, 137–141.
12. OVSYANNIKOV, L. V., On the property of x-autonomy. *Dokl. Ross. Akad. Nauk*, 1993, **330**, 5, 559–561.

13. OVSYANNIKOV, L. V., On the hierarchy of invariant submodels of differential equations. *Dokl. Ross. Akad. Nauk*, 1998, **361**, 6, 740–742.
14. OVSYANNIKOV, L. V. and CHUPAKHIN, A. P., Regular partly invariant submodels of the gas dynamics equations. *Prikl. Mat. Mekh.*, 1996, **60**, 6, 990–999.
15. OVSYANNIKOV, L. V., Regular type (2, 1) submodels of the gas dynamics equations. *Zh. Prikl. Mekh. Tekh. Fiz.*, 1996, **37**, 2, 3–13.
16. CHUPAKHIN, A. P., *Non-barochronous submodels of type (1, 2) and (1, 1) of the gas dynamics equations*. Preprint No. 5–98. Inst. Gidrodinamiki, Sib. Otd. Ross. Akad. Nauk, Novosibirsk, 1998.
17. SIDOROV, A. F., SHAPEYEV, V. P. and YANENKO, N. N., *The Method of Differential Relations and its Applications in Gas Dynamics*. Nauka, Novosibirsk, 1984.
18. MELESHKO, S. V., Non-isentropic steady-state spatial and plane unsteady double waves. *Prikl. Mat. Mekh.*, 1989, **53**, 2, 255–260.
19. MELESHKO, S. V., Group classification of the equations of the gas motions in a constant force field. *Zh. Prikl. Mekh. Tekh. Fiz.*, 1996, **37**, 1, 42–47.
20. OVSYANNIKOV, L. V., Isobaric gas motions. *Differents. Uravneniya*, 1994, **30**, 10, 1792–1799.
21. MELESHKO, S. V., Group classification of the equations of two-dimensional gas motions. *Prikl. Mat. Mekh.*, 1994, **58**, 4, 56–62.
22. KHABIROV, S. V., Submodel of spiral motions in gas dynamics. *Prikl. Mat. Mekh.*, 1996, **60**, 1, 53–65.
23. KHABIROV, S. V., Spiral motions in gas dynamics with pressure and density solely dependent on time. *Mat. Zametki*, 1996, **59**, 1, 133–141.
24. KHABIROV, S. V., Submodel of the rotational gas motions in a uniform force field, *Prikl. Mat. Mekh.*, 1998, **62**, 2, 263–271.
25. CHUPAKHIN, A. P., The barochronous gas motions. *Dokl. Ross. Akad. Nauk*, 1997, **352**, 5, 624–626.
26. CHUPAKHIN, A. P., *Barochronous gas motions. General properties and submodels of type (1.2) and (1. 1)*. Preprint No. 4–98. Inst. Gidrodinamiki, Sub. Otd. Ross. Akad. Nauk, 1998.
27. OVSYANNIKOV, L. V., Singular vortex. *Zh. Prikl. Mekh. Tekh. Fiz.*, 1995, **36**, 3, 45–52.
28. OVSYANNIKOV, L. V., Plane gas flows with closed streamlines. *Dokl. Ross. Akad. Nauk*, 1998, **361**, 1, 51–53.
29. MELESHKO, S. V., On one class of partially invariant solutions of plane gas flows. *Differents. Uravneniya*, 1994, **30**, 10, 1825–1827.
30. GOLOVIN, S. V., On an invariant solution of the gas dynamics equations. *Zh. Prikl. Mekh. Tekh. Fiz.*, 1997, **38**, 1, 3–10.
31. MAMONTOV, Ye. V., Invariant, rank two submodels of the gas dynamics equations. *Zh. Prikl. Mekh. Tekh. Fiz.*, 1999, **40**, 2, 50–55.

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